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Effect of radiative heat loss on steady hypersonic flow past a blunt body

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Using the grey gas approximation, the effect of radiative heat loss on axially symmetric flows is studied. Using an expansion procedure about the axis of symmetry, a numerical solution for the stagnation region is found taking the shock to be spherical. The results of this calculation are compared with the results of Lighthill's non-radiative constant density solution.

1. Introduction

In hypersonic flow past a blunt body there appears in front of the body a strong shock across which the kinetic energy of the incident stream is greatly reduced. As a result the temperature of the gas increases, and for sufficiently large values of the mainstream velocity, the loss of energy by radiation becomes important.

Goulard (1962) has shown by a dimensional analysis of the shock layer that two parameters determine the nature of the flow. These are the characteristic optical length τ_0 of the system, defined by

$$\tau_0 = k_0 L,$$

where k_0 denotes a typical value of the volumetric absorption coefficient of the gas and L is a typical length, and the radiation convection ratio Γ. When τ_0 is small, that is when the amount of radiation absorbed by an element of gas from the surrounding gas is small compared with the amount of radiation emitted by this element, Γ is defined by

$$\Gamma = \frac{4\sigma k_0 L T_0^4}{\rho_0 u_0 i_0},$$

where the suffix 0 indicates some arbitrary reference state. The quantities ρ, T, u, i and σ denote density, temperature, velocity, enthalpy and the Stefan Boltzmann constant respectively.

For the hypersonic detached shock layer the most convenient reference state to choose is that of the gas immediately behind the shock for the radiationless solution, taking L to be the stand-off distance Δ. In this case Γ is given by

$$\Gamma = \frac{4\sigma k_s \Delta T_s^4}{\frac{1}{2}\rho_\infty U_\infty^3},$$

where the suffix ∞ refers to the uniform conditions in the main-stream.

Goulard (1964) has considered the cases when Γ is sufficiently small for the radiationless solution to apply, and $\Gamma \ll 1$ when a perturbation scheme is adequate. Kennet (1962) has argued from the results of Bird (1960) that the introduction of a radiation loss term into the energy equation has little effect on the velocity distribution in the shock layer. Using Lighthill's (1957) constant density solution for the velocity distribution he determined the radiation lost by the gas. However, it seems that this approach can be considered only for $\Gamma \sim 0$.

In the problem considered here, the gas is assumed to be optically thin while the radiation-convection ratio Γ is taken to be of order unity. Goulard (1961) has shown that such flight conditions as these might be encountered by a 'Mars probe' re-entering the earth's atmosphere. The gas is assumed to be perfect and the flow is axially symmetric although the analysis could easily be extended to include plane symmetric flows. A solution for the stagnation region is presented.

The gas is assumed to be grey. This means that the volumetric absorption coefficient is taken to be independent of the frequency of the radiation. In general this is an approximation which, for practical purposes, must be regarded with suspicion. However, in this case it will be seen that the radiative heat loss term is of the form found empirically by Thomas (1962).

In the stagnation region the shape of the shock is assumed to be spherical. The absorption by the gas of radiation from the body surface is included in the analysis, and the resulting differential equations are solved numerically. The solution is presented in graphical form in §4 and compared with the radiationless solution of Lighthill.

2. The equations of motion

It is assumed that the radiating gas behaves like a perfect gas so that the equation of state takes the form

$$P = \rho T, \tag{2.1}$$

using a suitable choice of units. Also, the specific enthalpy i is given by

$$i = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}, \tag{2.2}$$

where γ is the ratio of specific heats of the gas and is assumed to be constant.

In practice, the gas will certainly not behave like a perfect gas. Goulard (1963) has considered an ideal planetary entry atmosphere not unlike the earth's atmosphere for altitudes less than 10^5 ft. In this model, the specific enthalpy was taken to be proportional to the square of the temperature, there being only a slight dependence on the density. Consequently, it can be seen from (3.7) that the effect of taking the gas to be perfect instead of behaving like the gas of Goulard's ideal atmosphere, is to increase the exponent of T by one. However, it can be seen from the results in §4 that variations with respect to β are of secondary importance compared with variations in Γ or the density ratio across the shock.

It has been shown (Goulard 1962; Chandrasekar 1960) that the effect of the gas radiating is to introduce three characteristic quantities of the radiation into the usual conservation equations of fluid dynamics. These three characteristic quantities are: the radiative stored energy U^R, the radiative pressure tensor p_{ij}^R, and the energy flux vector $\mathbf{F}^R$. However, Goulard (1962) has found that for most aerodynamic purposes U^R and p_{ij}^R are negligible compared with the internal energy per unit volume of the gas and the usual hydrodynamic pressure tensor p_{ij}.

Hence, assuming that the gas is inviscid, the momentum equation may be written

$$\frac{D\mathbf{v}}{Dt}+\frac{1}{\rho}\nabla p = 0, \tag{2.3}$$

where $\mathbf{v}$ is the velocity. The continuity equation for steady flow is

$$\nabla . (\rho \mathbf{v}) = 0, \tag{2.4}$$

and the energy equation is

$$\rho\frac{Di}{Dt}-\frac{Dp}{Dt} = -\nabla . \mathbf{F}^R. \tag{2.5}$$

The gas is assumed to be 'grey' and non-scattering and to be in local thermodynamic and chemical equilibrium. The surface of the body is assumed to radiate like a black body and to absorb all the radiation incident on it. Under these assumptions the divergence of the radiation flux vector becomes, using Goulard's notation,

$$\nabla . \mathbf{F}^R = \pi k(M)\left[4B(M)-\frac{1}{\pi}\int_{4\pi}\int_Q^M k(P)\,B(P)\exp\{-\tau_{MP}^s\}d\omega\, ds - \frac{1}{\pi}\int_{4\pi} B(Q)\exp\{-\tau_{MQ}^s\}d\omega\right], \tag{2.6}$$

where B is the Planck function defined by

$$B = \sigma T^4/\pi, \tag{2.7}$$

and M is the point of interest in the gas. The line MP cuts either the body or the shock in Q. Also, τ_{MP}^s and τ_{MQ}^s are optical lengths based on MP and MQ respectively.

It can be seen that the three terms on the right-hand side of (2.6) represent the energy per unit volume emitted by an element of gas at M, the energy absorbed by this element from the surrounding gas and from the surface of the body.

If the gas is assumed to be optically thin, the second term on the right-hand side of (2.6) can be neglected in comparison with the first. Also, the third term may be taken as $2B(Q)$ for the stagnation region, assuming that the temperature of the body surface is constant and that the shock layer is thin compared with the dimensions of the body, there being no source of radiation ahead of the shock.

Using available numerical data Thomas (1962) found that, for air, the rate of heat loss per unit volume due to radiation was proportional to $\rho^\alpha T^\beta$ where $\alpha = 1{\cdot}28$ and $\beta = 10{\cdot}54$. Consequently, the volumetric absorption coefficient for this problem is taken to be of the form

$$k \propto \rho^\alpha T^{\beta-4}. \tag{2.8}$$

Hence

$$\nabla . \mathbf{F}^R = R'\rho^\alpha T^{\beta-4}(T^4-C^4), \tag{2.9}$$

where R' and C are constants and C is $2^{-\frac{1}{4}}$ times the temperature of the surface of the body.

It is usual to assume that the temperature of the body is so low that the third term on the right-hand side of (2.6) can be neglected. However, in this treatment of the problem the term is retained since no other mechanism for supplying the gas with energy is considered. Hence, at the stagnation point where the gas is stationary the temperature would drop unrealistically if the term were omitted.

If the suffix s denotes conditions immediately behind the shock and the suffix ∞ refers to the uniform conditions in the mainstream, the usual strong shock conditions for hypersonic flow are

$$\frac{\rho_\infty}{\rho_s} = \frac{\gamma-1}{\gamma+1}, \tag{2.10}$$

$$\left.\begin{aligned} u_{ns} &= \frac{\gamma-1}{\gamma+1} U_\infty \sin\Theta, \\ u_{ts} &= U_\infty \cos\Phi, \\ p_s &= \frac{2}{\gamma+1}\rho_\infty U_\infty^2 \sin^2\Theta \\ \text{and} \qquad i_s &= \frac{2\gamma}{(\gamma+1)^2} U_\infty^2 \sin^2\Theta. \end{aligned}\right\} \tag{2.11}$$

The quantities u_n and u_t are the velocity components normal and tangential to the shock surface respectively, and Θ is the inclination of the shock to the mainstream.

3. Method of solution

The equations are to be solved in the neighbourhood of the front stagnation point. This region is not only important from the point of view of radiative heat transfer but also a solution for this region will determine the shock stand-off distance, one of the more easily measurable quantities in hypersonic flow past a blunt body. There will be no great loss of generality if the shock is assumed to be spherical in the region of interest. This assumption also has the advantage that the solution can be compared with that which Lighthill obtained for a non-radiating gas.

The non-dimensional flow variables are denoted by a prime and are defined by

$$\mathbf{v}' = \mathbf{v}/U_\infty, \quad p' = p/\rho_\infty U_\infty^2, \quad T' = T/U_\infty^2,$$
$$\rho' = \rho/\rho_\infty, \quad \psi' = \psi/\rho_\infty U_\infty R_s^2$$

and $\boldsymbol{\omega}' = \boldsymbol{\omega} R_s/U_\infty$, where R_s is the radius of curvature of the shock and $\boldsymbol{\omega}$ is the vorticity, ψ being a stream function.

Spherical polar co-ordinates (r, θ, λ) are used with the line $\theta = 0$ pointing upstream so that for axially symmetric flow there is no λ dependence. The length r is non-dimensionalized, hence

$$r' = r/R_s,$$

and the origin of the co-ordinate system is chosen so that the shock is given by $r' = 1$.

There is no confusion if the primes which denote non-dimensionalized quantities are henceforward omitted.

The equation of motion (2.3) may be written in the invariant form,

$$\rho(\boldsymbol{\omega}\wedge\mathbf{v})+\rho\nabla(\tfrac{1}{2}\mathbf{v}^2) = -\nabla p, \tag{3.1}$$

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$. Since the flow is axially symmetric $\omega_r = \omega_\theta = 0$, and

$$\omega_\lambda = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial \theta}, \tag{3.2}$$

where the suffices r, θ and λ refer to the r, θ and λ components.

The pressure term can be eliminated from (3.1) simply by taking the curl of the equation, and the result is

$$\operatorname{curl}[\rho(\boldsymbol{\omega}\wedge\mathbf{v})]+\operatorname{curl}[\rho\nabla(\tfrac{1}{2}\mathbf{v}^2)] = 0.$$

In the co-ordinate system chosen, this becomes

$$\omega_\lambda v_r\frac{\partial\rho}{\partial r}+\omega_\lambda\frac{v_\theta}{r}\frac{\partial\rho}{\partial\theta}+\frac{\rho}{r}\frac{\partial}{\partial r}(r\omega_\lambda v_r)+\frac{\rho}{r}\frac{\partial}{\partial\theta}(\omega_\lambda v_\theta)$$
$$= -\frac{\partial\rho}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}[\tfrac{1}{2}(v_r^2+v_\theta^2)]+\frac{1}{r}\frac{\partial\rho}{\partial\theta}\frac{\partial}{\partial r}[\tfrac{1}{2}(v_r^2+v_\theta^2)]. \tag{3.3}$$

Also, the continuity equation (2.4) is

$$\frac{\partial}{\partial r}(\rho r^2\sin\theta v_r)+\frac{\partial}{\partial\theta}(\rho r\sin\theta v_\theta) = 0, \tag{3.4}$$

and, using this, (3.3) may be simplified to give

$$\left(v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\omega_\lambda}{\rho r^2\sin\theta}\right) = \frac{1}{\rho r^2\sin\theta}\left(\frac{\partial\rho}{\partial\theta}\frac{\partial}{\partial r}-\frac{\partial\rho}{\partial r}\frac{\partial}{\partial\theta}\right)[\tfrac{1}{2}(v_r^2+v_\theta^2)]. \tag{3.5}$$

In addition, the r-component of (3.1) is used, that is,

$$-\omega_\lambda v_\theta+\frac{\partial}{\partial r}[\tfrac{1}{2}(v_r^2+v_\theta^2)] = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{3.6}$$

Using the relations (2.2) and (2.9) the energy equation (2.5) becomes

$$\rho\left(v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)-\left(v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}\right)p = -\frac{\gamma}{\gamma-1}R'\rho^\alpha T^{\beta-4}(T^4-C^4), \tag{3.7}$$

where R' is a non-dimensional parameter corresponding to R and in fact

$$\Gamma = \left(\frac{\gamma-1}{2}\right)^\beta\left(\frac{\gamma-1}{\gamma+1}\right)^{-\alpha}\frac{\gamma}{\gamma+1}R',$$

taking Δ for the radiationless solution to be $(\gamma-1)R_s/(\gamma+1)$.

A streamfunction, ψ is introduced to satisfy the continuity equation (3.4). Hence ψ is defined by

$$\left.\begin{aligned}\frac{\partial\psi}{\partial r} &= \rho r\sin\theta v_\theta,\\ \text{and}\qquad \frac{\partial\psi}{\partial\theta} &= -\rho r^2\sin\theta v_r.\end{aligned}\right\} \tag{3.8}$$

In principle, (3.5) to (3.8) are soluble using (3.2) for ω_λ and (2.1) to relate p, ρ and T.

The shock conditions (2.11) become

$$\left.\begin{aligned} v_{rs} &= -\frac{\gamma-1}{\gamma+1}\cos\theta, \\ v_{\theta s} &= \sin\theta, \\ P_s &= \frac{2}{\gamma+1}\cos^2\theta \\ \text{and} \qquad T_s &= \frac{2(\gamma-1)}{(\gamma+1)^2}\cos^2\theta. \end{aligned}\right\} \tag{3.9}$$

In the neighbourhood of the stagnation region θ is small, and hence an expansion of the flow variables in terms of θ is used. This expansion procedure has the advantage that a solution for the leading terms will give the value of the shock stand-off distance. Also, variations with respect to θ are small compared with variations with respect to r, so that the leading terms should give an adequate solution small distances from the axis.

It follows from the symmetry of the flow that the relevant expansions are

$$\left.\begin{aligned} T &= T_0(r) + O(\theta^2), \\ \rho &= \rho_0(r) + O(\theta^2), \\ P &= P_0(r) + O(\theta^2) \\ \text{and} \qquad \psi &= \psi_0(r)\theta^2 + O(\theta^4). \end{aligned}\right\} \tag{3.10}$$

It can be seen from (3.8) that the components of velocity may be expressed in the form

$$v_\theta = \frac{1}{\rho_0 r}\frac{d\psi_0}{dr}\theta + O(\theta^3) \tag{3.11}$$

and

$$v_r = -\frac{2\psi_0}{\rho_0 r^2} + O(\theta^2). \tag{3.12}$$

Hence, using (3.2), ω_λ is given by

$$\omega_\lambda = \frac{1}{r}\frac{d}{dr}\left(\frac{1}{\rho_0}\frac{d\psi_0}{dr}\right)\theta + O(\theta^3), \tag{3.13}$$

where it has been assumed that the rates of change with respect to r are much greater than those with respect to θ.

Using (3.10) to (3.13), equation (3.5) may be written

$$\frac{2\psi_0}{\rho_0 r^2}\frac{d}{dr}\left[\frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{\rho_0}\frac{d\psi_0}{dr}\right)\right] = \frac{1}{\rho_0^3 r^4}\frac{d\rho_0}{dr}\left(\frac{d\psi_0}{dr}\right)^2, \tag{3.14}$$

where approximations consistent with those in equation (3.13) have been made.

Also, (3.6) and (3.7) become

$$\frac{2\psi_0}{\rho_0 r^2}\frac{d}{dr}\left(\frac{2\psi_0}{\rho_0 r^2}\right) = -\frac{dT_0}{dr} - \frac{T_0}{\rho_0}\frac{d\rho_0}{dr} \tag{3.15}$$

and

$$-\frac{2\psi_0}{r^2}\frac{\gamma}{\gamma-1}\frac{dT_0}{dr} + \frac{2\psi_0}{r^2}\left(\frac{dT_0}{dr} + \frac{T_0}{\rho_0}\frac{d\rho_0}{dr}\right) = -\frac{\gamma}{\gamma-1}R'\rho_0^\alpha T_0^{\beta-4}(T_0^4 - C^4), \tag{3.16}$$

using equation (2.1).

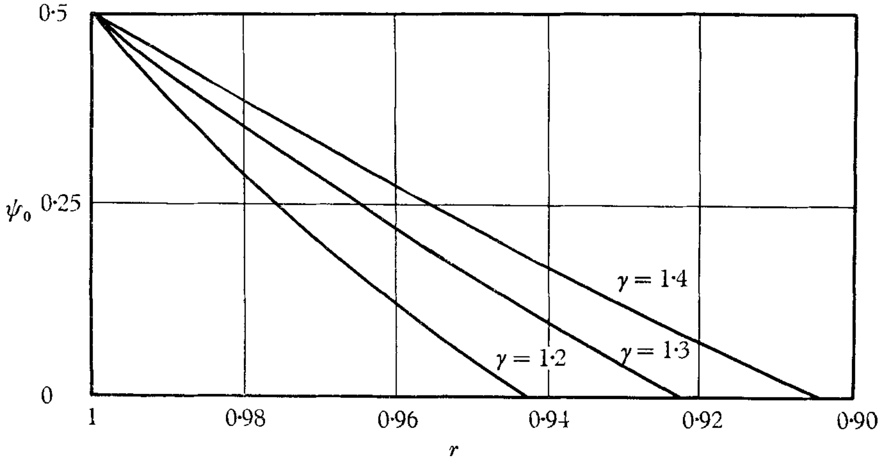

FIGURE 1. ψ_0 as a function of r for various values of γ with $A = \frac{1}{2}$ and $\beta = 10$.

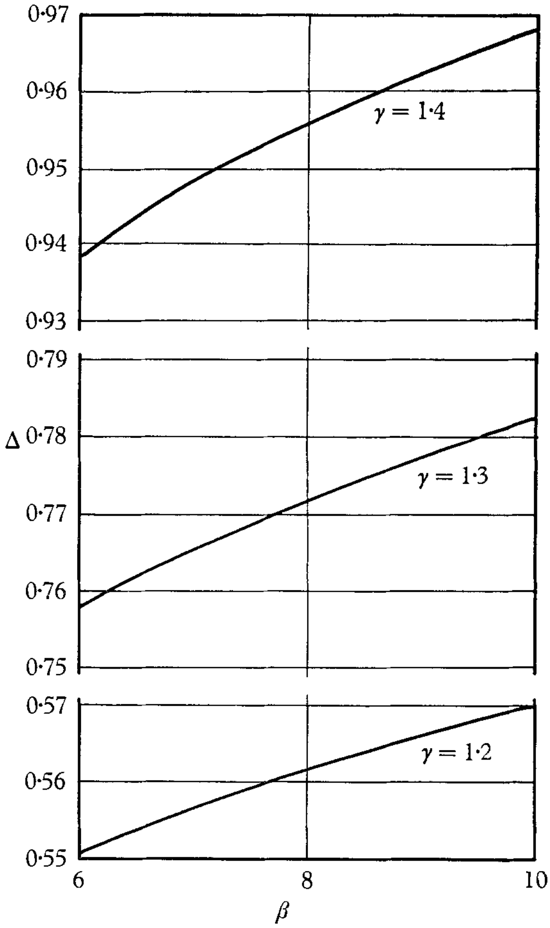

FIGURE 2. The stand-off distance, Δ, as a function of β for various values of γ with $A = \frac{1}{2}$.

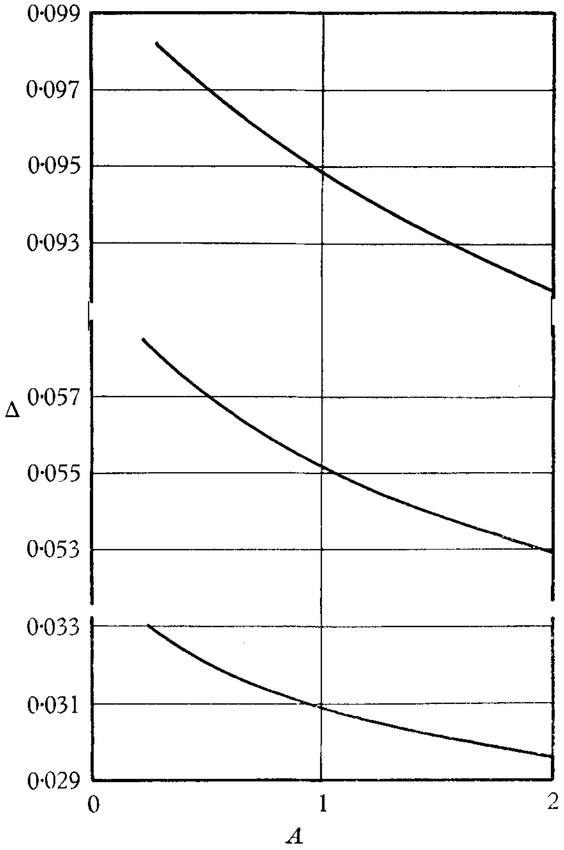

FIGURE 3. The stand-off distance, Δ, as a function of A for $\gamma = 1{\cdot}1$, $1{\cdot}2$ and $1{\cdot}4$ with $\beta = 10$.

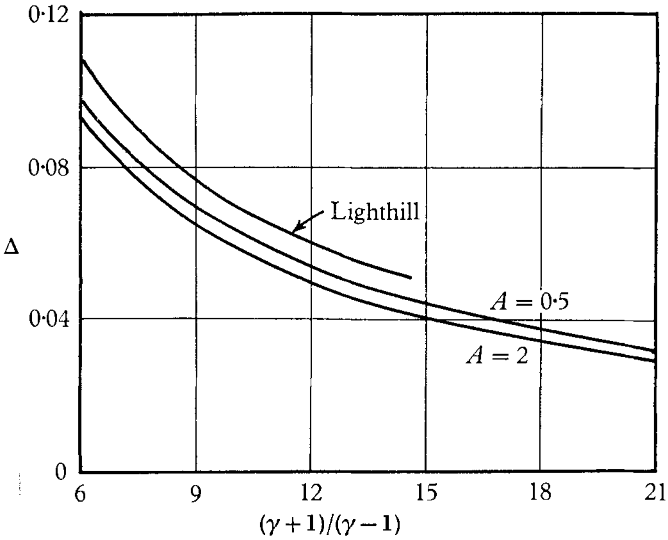

FIGURE 4. The stand-off distance, Δ, as a function of $(\gamma+1)/(\gamma-1)$ for various values of A with $\beta = 10$, compared with the radiationless solution of Lighthill (1957).

These three equations for ψ_0, ρ_0 and T_0 are now expressed as five first-order differential equations and using the conditions at the shock are solved numerically on the KDF 9 computer by the Runge Kutta Merson method.

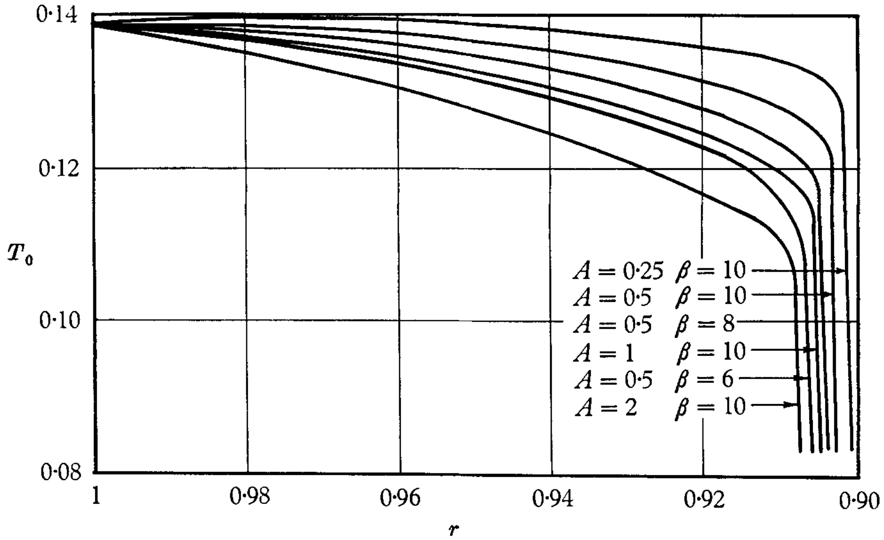

FIGURE 5. The temperature distribution on the axis of symmetry, $T_0(r)$, for various values of β and A with $\gamma = 1{\cdot}4$.

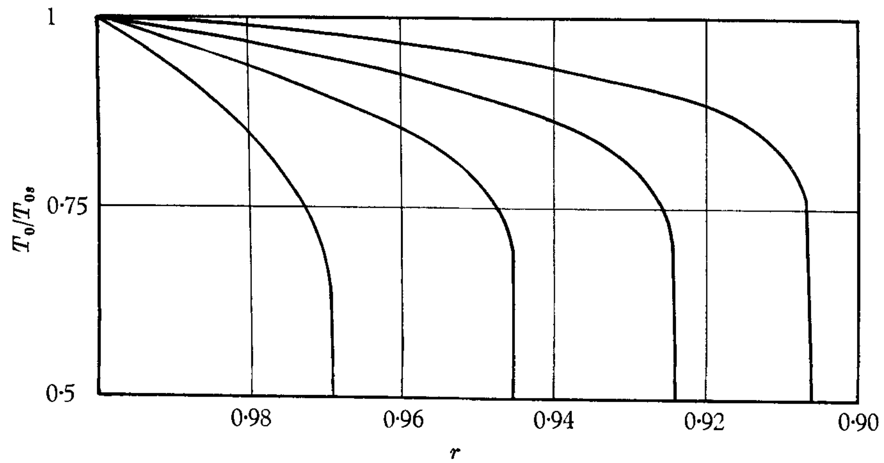

FIGURE 6. $T(r)/T(1)$ plotted for $\gamma = 1{\cdot}1$, $1{\cdot}2$, $1{\cdot}3$ and $1{\cdot}4$ with $A = 0{\cdot}5$ and $\beta = 6$.

4. The numerical results

The equations (3.14) to (3.16) are solved for $C = \frac{1}{2}T_{0s}$, $\alpha = 1$ and various values of β and A where A is given by

$$R' = A\left(\frac{\gamma-1}{\gamma+1}\right)^{-\beta+1}. \tag{4.1}$$

Hence, if A is $O(1)$, the radiation convection ratio is also $O(1)$.

In figure 1, ψ_0 is plotted against r for different values of γ with $A = \frac{1}{2}$ and $\beta = 10$. From this the shock stand-off distance can be determined, that is, the value of $1-r$ for which $\psi_0 = 0$. The shock stand-off distance is also evaluated for other values of A and β. The variations with β and with A for constant γ are shown in figures 2 and 3 respectively. In figure 4, the shock stand-off distance is shown as a function of γ for constant A and β and compared with the results of Lighthill.

The effect of allowing radiative heat loss is to reduce the stand-off distance. This is simply because the velocities in the stagnation region are small so that the pressure variations are correspondingly small; hence, the fall in temperature due to the heat lost produces a rise in the density. The rise in the density reduces the stand-off distance which is essentially determined by the mass flow.

The temperature distribution on the axis of symmetry is shown as a function of A, β and γ in figures 5 and 6.

The very rapid decrease in the temperature of the gas near the body is due to the fact that the surface temperature of the body was chosen to be significantly less than the temperature of the gas immediately behind the shock, and that heat conduction has been neglected. In practice this very rapid rate of change of temperature is smoothed out across a boundary layer.

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